SAT-Based Model Checking

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Outline

1. A Short Intro to Model Checking
   - Structures
   - Properties

2. SAT Solver Interface
   - To The Solver
   - From The Solver

3. Checking Invariants
   - Bounded Model Checking
   - Interpolation
   - Proving Invariants by Induction
   - IC3: Incremental Inductive Verification

4. Progress Properties and Branching Time
   - Bounded Model Checking
   - Incremental Inductive Verification (FAIR and $k$-Liveness)
   - Model Checking CTL
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Simple Synchronous Arbiter (Verilog)

module arbsim (input clock, input [1:2] r, output reg [1:2] g);
    initial g <= 0;
    always @ (posedge clock) begin
    end
endmodule // arbsim
Mutual Exclusion for the Simple Arbiter

\[ I(g) = \neg g_1 \land \neg g_2 \]

\[ \exists r_1, r_2. \ T(r, g, g') = \neg g_1' \lor \neg g_2' \]

\[ P(g) = \neg g_1 \lor \neg g_2 \]
The Model Checking Question

- Given a structure $S$ and a property $\varphi$, is $S$ a model of $\varphi$?
- Written $S \models \varphi$
- More in detail: does $\varphi$ hold for all computations of $S$?
  - From all initial states
Finite-State Transition Systems

Symbolic representation of a system:

\[ S : (\bar{i}, \bar{x}, I(\bar{x}), T(\bar{i}, \bar{x}, \bar{x}')) \]

- \( \bar{i} \): primary inputs
- \( \bar{x} \): state variables
- \( \bar{x}' \): next state variables
- \( I(\bar{x}) \): initial states
- \( T(\bar{i}, \bar{x}, \bar{x}') \): transition relation

\( I \) and \( T \) define a finite transition structure (Kripke structure)

- Every valuation of \( \bar{x} \) is a state
- \( \exists \bar{i}. T(\bar{i}, \bar{x}, \bar{x}') = T(\bar{x}, \bar{x}') \) defines the transitions
Composition

- Complex systems are composed of several modules
- Each module is described as a finite state structure $S_i$
- The overall Kripke structure is obtained as the product of the structures
  - State explosion!
- The product can be either synchronous or asynchronous (interleaving)
Examples of Temporal Logic Properties

- **G p**: $p$ is **invariably** true (always along all paths)
  - $p$ is an **atomic proposition**
  - G is a **temporal operator**

- **F p**: $p$ is **inevitably** true (sometimes true along all paths)

- **$p \, U \, q****: $q$ eventually holds and $p$ holds **until** then

- **G($p \rightarrow X \, q$)**: every $p$ is immediately followed by a $q$
  - Only allowed if time is discrete

- **G F($p \rightarrow q$)**: if $p$ is persistent, then $q$ is inevitable
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Properties

- Properties are sets of behaviors
- Various specification mechanisms are in use: Temporal logics and automata are popular
- The examples we have seen are formulae of the temporal logic LTL (Linear-Time Logic)
- Syntactic sugar often useful (e.g., PSL, Property Specification Language)
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Linear time logics reason about sets of computation paths
Branching Time

Branching time logics reason about computation trees
Invariance, Safety, and Progress

- **Invariance** properties say that certain states are unreachable
  - Reachability analysis

- **Safety** properties say that certain events never happen
  - Generalize invariants and can be reduced to them

- **Progress** properties are the non-safety properties
  - Cycle detection (for finite state systems)

- **Liveness** (Alpern and Schneider [1985]) is related to progress, but not the same

- This can be made (a lot) more formal
  - Why is $G(p \rightarrow X q)$ a safety property, but $G(p \rightarrow F q)$ is not?
  - Borel hierarchy, (Landweber [1969])
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Automata

- Properties may be described by automata that take the computation of the system as input and either accept it or reject it.
- For non-terminating computations and linear-time properties we need $\omega$-automata, which accept $\omega$-regular languages.
- For linear-time model checking we need the automaton for the negation of the property of interest.
  - Model checking reduced to checking language emptiness of an $\omega$-automaton.
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Omega-Automata

- $\omega$-automata describe linear-time properties
  - Nondeterministic Büchi automata recognize all $\omega$-regular properties
- Examples of Büchi automata (an accepting run visits some accepting state infinitely often)

- They are more expressive than LTL
From Formula to Büchi Automaton

\[ \psi \mathsf{U} \varphi = \varphi \lor [\psi \land X(\psi \mathsf{U} \varphi)] \]

- Expansion produces a DNF whose every term is the conjunction of:
  1. a propositional formula that must hold now and
  2. a temporal formula that must hold from the next step
Intro to MC
Solver Interface Invariants Beyond Safety

Branching Time Temporal Logic

- Add **path** quantifiers to LTL to obtain CTL*
  - A: for all paths
  - E: for at least one path
- AG EF $p$: resetability
- LTL is embedded in CTL* by prepending A to all formulae
  - $AG(p \rightarrow F q)$
- $AG(p \rightarrow F q)$ is equivalent to $AG(p \rightarrow AF q)$, but…
- AF AG $p$ is not equivalent to AF G $p$
- Maidl [2000] for more info
- In CTL every temporal operator must be immediately preceded by a path quantifier
  - AG $\phi$, A $\psi$ U $\phi$, AF $\phi$, AX $\phi$, EG $\phi$, E $\psi$ U $\phi$, EF $\phi$, EX $\phi
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Linear vs. Branching Time

- Branching time is more powerful, but less intuitive
  - Resetability
  - $A F G \varphi$ vs. $AF AG \varphi$
- Structure equivalence is finer-grained for branching time:
  - Linear time $\leftrightarrow$ language (trace) equivalence
  - Branching time $\leftrightarrow$ simulation relations
- Linear time is more suitable for compositional verification and Bounded Model Checking
- Counterexample generation simpler for linear time
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   - Model Checking CTL
## From Hardware Description Language to CNF

- **From source code to CDFG**
- **From CDFG to formulae over bit vectors and finite-domain variables**
  - May involve abstraction
- **Bit-blasting (binary encoding) to Boolean circuit plus memory elements**
- **Optimization of Boolean circuit**
  - Often uses And-Inverter Graphs (AIGs) or similar data structures
- **Conversion of circuit to CNF**
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Distributivity

- Apply \((a \land b) \lor c = (a \lor c) \land (b \lor c)\) systematically along with simplifications
- Preserves equivalence and does not introduce new variables
- Size may blow up
  - \((a \land b) \lor (c \land d) = (a \lor c) \land (a \lor d) \land (b \lor c) \land (b \lor d)\)
  - \((x_1 \land x_2 \land x_3) \lor (x_4 \land x_5 \land x_6) \lor \cdots\)
- Seldom applied in its pure form
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Equisatisfiability

Two formulae $F$ and $G$ are equisatisfiable if

1. $F$ is satisfiable iff $G$ is satisfiable.

2. If $\eta_F$ ($\eta_G$) is a satisfying assignment for $F$ ($G$), there exists a satisfying assignment $\eta_G$ ($\eta_F$) for $G$ ($F$) that agrees with $\eta_F$ ($\eta_G$) on all the variables that $F$ and $G$ have in common.

A common case occurs when one of the two formulae, say $G$, contains all the variables in the other formula. Then a satisfying assignment for $F$ can be easily derived from one for $G$ by dropping the extra variables.
Tseitin

Use definitions for subformulae

\[ f \leftrightarrow g \lor h \]
\[ g \leftrightarrow a \land b \]
\[ h \leftrightarrow c \land d \]

Then, from \((a \land b) \lor (c \land d)\), we get

\[ (a \lor \neg g) \land (b \lor \neg g) \land (\neg a \lor \neg b \lor g) \]
\[ \land (c \lor \neg h) \land (d \lor \neg h) \land (\neg c \lor \neg d \lor h) \]
\[ \land (\neg g \lor f) \land (\neg h \lor f) \land (g \lor h \lor \neg f) \land f \]
Simpler Equisatisfiable CNF Formulae

If the formula is in negation normal form, Tseitin’s translation can be simplified (Plaisted and Greenbaum [1986])

\[
\begin{align*}
f & \rightarrow g \lor h \\
g & \rightarrow a \land b \\
h & \rightarrow c \land d
\end{align*}
\]

Then, from \((a \land b) \lor (c \land d)\), we get

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\end{align*}
\]
More Conversions to CNF

- Wilson, Sheridan
- Nice DAGs
- Cut-based
- BDD-based
- SAT preprocessor
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Proofs of Unsatisfiability

Different verification techniques require

- Resolution proofs
- UNSAT cores
- Assumptions (unit clauses) in UNSAT cores
  - Can be extracted with minimal overhead (Eén and Sörensson [2003])
Incremental Solving

- Solve sequences of related SAT instances
- Ability to push and pop clauses (efficiently)
- Keep learned clauses that are still valid
  - All learned clauses remain valid if no clause is popped
- Keep variable scores
- Multiple solver objects
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- Based on unrolling the transition relation
- Looks for counterexamples of certain lengths
- May be extended to a complete method
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- Checks for a counterexample to a property of a model
  - We assume finite state
- Encodes the property checking problem as propositional satisfiability (SAT)
- Constructs a propositional formula that is satisfiable iff there exits a length-$k$ counterexample, e.g.,
  \[ I(\bar{x}_0) \land \bigwedge_{0 \leq i < k} T(\bar{i}_i, \bar{x}_i, \bar{x}_{i+1}) \land \neg P(\bar{x}_k) \]
- If no counterexample is found, BMC increases $k$ until
  - a counterexample is found,
  - the search becomes intractable, or
  - $k$ reaches a certain bound
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$$I(x_0) \land \bigwedge_{0 \leq i < k} T(i, x_i, x_{i+1}) \land \neg P(x_k)$$

- If no counterexample is found, BMC increases $k$ until
  - a counterexample is found,
  - the search becomes intractable, or
  - $k$ reaches a certain bound
Bounded Model Checking

- Checks for a counterexample to a property of a model
  - We assume finite state
- Encodes the property checking problem as propositional satisfiability (SAT)
- Constructs a propositional formula that is satisfiable iff there exists a length-$k$ counterexample, e.g.,

$$I(\bar{x}_0) \land \bigwedge_{0 \leq i < k} T(\bar{i}_i, \bar{x}_i, \bar{x}_{i+1}) \land \neg P(\bar{x}_k)$$

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Proving Properties with BMC

- The original BMC algorithm (Biere et al. [1999]), although complete for finite state, is limited in practice to falsification.

- BMC can prove that an invariant \( \psi \) holds on a model \( S \) only if a bound, \( \kappa \), is known such that:
  - if no counterexample of length up to \( \kappa \) is found, then \( S \models \psi \)

- Several methods exist to compute a suitable \( \kappa \)

- The optimum value of \( \kappa \), however, is usually very expensive to obtain

  - Finding it is at least as hard as checking whether \( S \models \psi \) (Clarke et al. [2004])
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Finding The Bound $\kappa$

- **Compute diameter** of graph
  - Minimum $d$ such that, if there is a path of length $d + 1$ between two states, then there is a path of length at most $d$ between the same states
  - $\forall x_0, \ldots, x_{d+1} \cdot \bigwedge_{0 \leq i \leq d} T(x_i, x_{i+1}) \Rightarrow \exists x'_0, \ldots, x'_{d} \cdot (\bigwedge_{0 \leq i < d} T(x'_i, x'_{i+1}) \land x'_0 = x_0 \land \bigvee_{0 \leq i \leq d} x'_i = x_{d+1})$

- If one end of the path is constrained to an initial (target) state, one obtains the forward (backward) recursive radius of the graph

- Restrict search to simple paths (next slide)
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A counterexample to an invariant is a finite prefix path to a state that satisfies $\neg P$ (bad state).

If a counterexample exists, then there is a simple path from an initial state to a bad state that goes through no other initial or bad state.

An invariant holds (Sheeran et al. [2000]) if:
- there is no counterexample of length $k$ to $\neg P$, and
- no simple path of length $k + 1$ to $\neg P$ that does not go through any other states satisfying $\neg P$, or
- no simple path of length $k + 1$ from an initial state that does not go through any other initial states.
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Checking for Simple Paths

- Simple-minded check produces quadratic formula
  \[ \bigwedge_{0<i\leq k} \bigwedge_{0\leq j<i} (\bar{x}_i \neq \bar{x}_j) \]

- Using a bitonic sorting network (Kröning and Strichman [2003]) reduces the complexity to \( O(k \log^2 k) \)

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$k$-Induction

- Sheeran et al. call their method $k$-induction
- If all states on length-$k$ paths from the initial states satisfy $p$, and
- $k$ consecutive states satisfying $p$ are always followed by a state satisfying $p$, then
- all states reachable from the initial states satisfy $p$
- The second premise is verified when there are no simple paths of length $k + 1$
Abstraction Refinement

- Assume abstract model $S_a$ and abstraction of property $\varphi_a$ such that $S_a \models \varphi_a$ implies $S \models \varphi$
  - Use complete method on abstract model $S_a$, but use BMC on the concrete model $S$ when a counterexample is found in $S_a$
    - Use the counterexample(s) found in $S_a$ to constrain search in $S$
    - If concretization fails, use UNSAT core to refine abstraction
    - One-to-one and one-to-many concretization possible
- It is possible to reverse the order: proof-based abstraction (Amla and McMillan [2004])
  - Use BMC and periodically extract abstract model from UNSAT core and check it with complete method
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Interpolation (McMillan [2003])

Suppose
\[ I(x_0) \land T(x_0, x_1) \land T(x_1, x_2) \land \cdots \land T(x_{k-1}, x_k) \land \neg P(x_k) \] is unsatisfiable

Let \( F_1 = I(x_0) \land T(x_0, x_1) \) and
\[ F_2 = T(x_1, x_2) \land \cdots \land T(x_{k-1}, x_k) \land \neg P(x_k) \]

Then \( F_1(x_0, x_1) \land F_2(x_1, \ldots, x_k) \) is unsatisfiable

Interpolant \( I_1(x_1) \) is such that
\[ I_1(x_1) \land F_2(x_1, \ldots, x_k) \] is unsatisfiable

\( I_1(x_1) \) can be computed in linear time from a resolution proof that \( F_1(x_0, x_1) \land F_2(x_1, \ldots, x_k) \) is unsatisfiable

\( \exists x_0 \cdot I(x_0) \land T(x_0, x_1) \) is the strongest interpolant

set of states reachable from \( I(x_0) \) in one step
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- Then \( F_1(x_0, x_1) \land F_2(x_1, \ldots, x_k) \) is unsatisfiable

- Interpolant \( I_1(x_1) \) is such that
  - \( F_1(x_0, x_1) \to I_1(x_1) \)
  - \( I_1(x_1) \land F_2(x_1, \ldots, x_k) \) is unsatisfiable

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Then \( F_1(\bar{x}_0, \bar{x}_1) \land F_2(\bar{x}_1, \ldots, \bar{x}_k) \) is unsatisfiable

- Interpolant \( \mathcal{I}_1(\bar{x}_1) \) is such that
  - \( F_1(\bar{x}_0, \bar{x}_1) \rightarrow \mathcal{I}_1(\bar{x}_1) \)
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- set of states reachable from \( I(x_0) \) in one step
Interpolation-Based Termination Check

- $\mathcal{I}_1(\overline{x}_1)$ is a superset of the states reachable in one step such that no member state has a path of length $k - 1$ to a bad state.
- Replace $I(\overline{x}_0)$ with $I(\overline{x}_0) \lor \mathcal{I}_1(\overline{x}_0)$ and repeat.
  - If formula still unsatisfiable, interpolant $\mathcal{I}_2(\overline{x}_1)$ is a superset of states reachable in one or two steps such that no member state has a path of length $k - 1$ to a bad state.
- A converging sequence of interpolants means that no states satisfying $\neg p$ (bad states) are reachable.
- At convergence, an inductive invariant is obtained.
- Convergence guaranteed when the backward recursive radius is reached.
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- A converging sequence of interpolants means that no states satisfying $\neg \neg p$ (bad states) are reachable
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- A converging sequence of interpolants means that no states satisfying $\neg p$ (bad states) are reachable.
- At convergence, an inductive invariant is obtained.
- Convergence guaranteed when the backward recursive radius is reached.
Let \( \text{Pre}(Q(\overline{x})) \) be the predicate describing the states that are predecessors of the states described by \( Q \).

- Repeated application of \( \text{Pre} \) from \( \neg P \) corresponds to backward breadth-first search from the error states.
  - It computes an inductive strengthening of the property.

- Common approach with BDDs.
- Can be adapted to CNF (McMillan [2002]).
  - Introduced the use of blocking clauses.
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Preimage Computation by Solution Enumeration

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- Common approach with BDDs.
- Can be adapted to CNF (McMillan [2002]).
  - Introduced the use of blocking clauses.
Back to The Simple Arbiter

\[ l(\overline{g}) = \neg g_1 \land \neg g_2 \]

\[ \exists r_1, r_2 . \ T(\overline{r}, \overline{g}, \overline{g}') = \neg g_1' \lor \neg g_2' \]

\[ P(\overline{g}) = \neg g_1 \lor \neg g_2 \]
Inductive Proofs for Transition Systems

- Prove **initiation** (base case)
  - $I(\bar{x}) \Rightarrow P(\bar{x})$
  - All initial states satisfy $P$
  - $(\neg g_1 \land \neg g_2) \Rightarrow (\neg g_1 \lor \neg g_2)$

- Prove **consecution** (inductive step)
  - $P(\bar{x}) \land T(\bar{i}, \bar{x}, \bar{x}') \Rightarrow P(\bar{x}')$
  - All successors of states satisfying $P$ satisfy $P$
  - $(\neg g_1 \lor \neg g_2) \land (\neg g_1' \lor \neg g_2') \Rightarrow (\neg g_1' \lor \neg g_2')$

- If both pass, all reachable states satisfy the property
  - $S \models P$
Visualizing Inductive Proofs

The inductive assertion (yellow) contains all initial (blue) states and no arrow leaves it (it is closed under the transition relation)
Counterexamples to Induction: The Troublemakers
Counterexamples to Induction: The Troublemakers
Invariant Strengthening
Invariant Strengthening
Invariant Strengthening
Invariant Strengthening
Strong and Weak Invariants

Induction is not restricted to:
- the strongest inductive invariant (forward-reachable states)
- ... or the weakest inductive invariant (complement of the backward-reachable states)
- \( \neg x_1 \) is simpler than \( \neg x_1 \land (\neg x_2 \lor \neg x_3) \) (strongest) and \( (\neg x_1 \lor \neg x_3) \) (weakest)
Completeness for Finite-State Systems

- CTIs are effectively bad states
  - If a CTI is reachable so is at least one bad state
- Remove CTI from $P$ and try again
- Eventually either:
  - An inductive strengthening of $P$ results
  - An initial state is removed from $P$
- In the latter case, a counterexample is obtained
Examples of Strengthening Strategies

- Removing one CTI at a time is very inefficient!
  - Several strategies in use to avoid that

- Fixpoint-based invariant checking: if $\nu Z . p \land AX Z$ converges in $n > 0$ iterations, then $\bigwedge_{0 \leq i < n} AX^i p$ is an inductive invariant
  - In fact, the weakest inductive invariant

- $k$-induction: if all states on length-$k$ paths from the initial states satisfy $p$, and $k$ distinct consecutive states satisfying $p$ are always followed by a state satisfying $p$, then all states reachable from the initial states satisfy $p$.

- Interpolation-based model checking: the converged interpolant is an inductive invariant

- fsis algorithm (Bradley and Manna [2007]): try to extract an inductive clause from CTI to exclude multiple CTIs
Relative Induction

\[ \varphi = \neg x_1 \land (x_1 \lor \neg x_2) \]
Relative Induction

¬x₁ is not inductive
Relative Induction

$x_1 \lor \neg x_2$ is inductive
\neg x_1 \text{ is inductive relative to } x_1 \lor \neg x_2
Shortcoming of Relative Induction

\[ P = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \]
\[ \varphi = (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \]
Shortcoming of Relative Induction

\[(x_1 \lor x_2) \land P \land T \not\Rightarrow (x'_1 \lor x'_2)\]
Shortcoming of Relative Induction

\[ (\neg x_1 \lor \neg x_2) \land P \land T \not\Rightarrow (\neg x'_1 \lor \neg x'_2) \]
Shortcoming of Relative Induction

\[(x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land P \land T \Rightarrow (x'_1 \lor x'_2) \land (\neg x'_1 \lor \neg x'_2)\]
Shortcoming of Relative Induction

\[(x_1 \lor x_2) \text{ and } (\neg x_1 \lor \neg x_2) \text{ are mutually inductive}\]
IC3: Basic Algorithm

IC3 (Bradley [2011]) stands for

1. Incremental Construction of
2. Inductive Clauses for
3. Indubitable Correctness

IC3 is an Incremental Inductive Verification (IIV) algorithm
Basic Tenets

- **Approximate reachability assumptions**
  - $F_i$: contains at least all the states reachable in $i$ steps or less
  - If $S \models P$, $F_i$ eventually becomes inductive for some $i$
  - Approximation is desirable: IC3 does not attempt to get the most precise $F_i$'s

- **Stepwise relative induction**
  - Learn useful facts via induction relative to reachability assumptions

- **Clausal representation**
  - Learn clauses (lemmas) from CTIs
  - A form of abstract interpretation
IC3 Invariants

- The four main invariants of IC3:
  
  \[
  I \Rightarrow F_0 \\
  F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \\
  F_i \Rightarrow P \quad 0 \leq i \leq k \\
  F_i \land T \Rightarrow F'_{i+1} \quad 0 \leq i < k
  \]

- Established if there are no counterexamples of length 0 or 1
- The implicit invariant of the outer loop: no counterexamples of length \( k \) or less
Reasonable Invariants

- $I \Rightarrow F_0$: $F_0$ overapproximates the initial condition. (In practice, $I = F_0$.)
- $F_i \Rightarrow F_{i+1}$: a state believed to be reachable in $i$ steps or less is also believed to be reachable in $i + 1$ steps or less
- $F_i \Rightarrow P$: no state believed to be reachable in $i$ steps or less violates $P$
- $F_i \land T \Rightarrow F'_{i+1}$: all the immediate successors of a state believed to be reachable in $i$ steps or less are believed to be reachable in $i + 1$ steps or less
Pseudo-Pseudocode

```c
bool IC3 {
    if (I \not\Rightarrow P \text{ or } I \land T \not\Rightarrow P')
        return \bot
    F_0 = I; F_1 = P; k = 1
    repeat {
        while (there are CTIs in F_k) {
            either find a counterexample and return \bot
            or refine F_1, \ldots, F_k
        }
        k ++
        set F_k = P and propagate clauses
        if (F_i = F_{i+1} for some 0 < i < k)
            return \top
    }
}
```
Example: Passing Property

No counterexamples of length 0 or 1

\[ I = \neg x_1 \land \neg x_2 \]
\[ P = \neg x_1 \lor x_2 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \]
\[ F_i \Rightarrow P \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_i \quad 0 \leq i < k \]
Example: Passing Property

Does $F_1 \land T \Rightarrow P'$?

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = P = \neg x_1 \lor x_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_i$

$0 \leq i < k$
Example: Passing Property

Found CTI $s = x_1 \land x_2$

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = P = \neg x_1 \lor x_2 \]

\[
\begin{align*}
  I & \Rightarrow F_0 \\
  F_i & \Rightarrow F_{i+1} \\
  F_i & \Rightarrow P \\
  F_i \land T & \Rightarrow F'_{i+1}
\end{align*}
\]

\[
\begin{align*}
  0 \leq i < k \\
  0 \leq i \leq k \\
  0 \leq i < k
\end{align*}
\]
Example: Passing Property

Is \( \neg s = \neg x_1 \lor \neg x_2 \) inductive relative to \( F_1 \)?

\[
F_0 = I = \neg x_1 \land \neg x_2 \\
F_1 = P = \neg x_1 \lor x_2 \\
I \Rightarrow F_0 \\
F_i \Rightarrow F_{i+1} \\
F_i \Rightarrow P \\
F_i \land T \Rightarrow F_{i+1}' \\
0 \leq i < k \\
0 \leq i \leq k \\
0 \leq i < k
\]
Example: Passing Property

No. Is $\neg s = \neg x_1 \lor \neg x_2$ inductive relative to $F_0$?

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = P = \neg x_1 \lor x_2$
Example: Passing Property

Yes. Generalize $\neg s$ at level 0 in one of the two possible ways: either $\neg x_1$ or $\neg x_2$

$$F_0 = I = \neg x_1 \land \neg x_2$$
$$F_1 = P = \neg x_1 \lor x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \quad 0 \leq i < k$$
$$F_i \Rightarrow P \quad 0 \leq i \leq k$$
$$F_i \land T \Rightarrow F'_{i+1} \quad 0 \leq i < k$$
Example: Passing Property

Update $F_1$

\[
F_0 = I = \neg x_1 \land \neg x_2 \\
F_1 = (\neg x_1 \lor x_2) \land \neg x_2
\]

\[
\begin{align*}
I &\Rightarrow F_0 \\
F_i &\Rightarrow F_{i+1} \\
F_i &\Rightarrow P \\
F_i \land T &\Rightarrow F'_{i+1}
\end{align*}
\]

\[
0 \leq i < k
\]

\[
0 \leq i \leq k
\]

\[
0 \leq i < k
\]
Example: Passing Property

No more CTIs in $F_1$. No counterexamples of length 2. Instantiate $F_2$

$$F_0 = I = \neg x_1 \land \neg x_2$$
$$F_1 = (\neg x_1 \lor x_2) \land \neg x_2$$
$$F_2 = P = \neg x_1 \lor x_2$$
Example: Passing Property

Propagate clauses from $F_1$ to $F_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = (\neg x_1 \lor x_2) \land \neg x_2$

$F_2 = (\neg x_1 \lor x_2) \land \neg x_2$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Passing Property

$F_1$ and $F_2$ are identical. Property proved

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = (\neg x_1 \lor x_2) \land \neg x_2 \]
\[ F_2 = (\neg x_1 \lor x_2) \land \neg x_2 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \]
\[ F_i \Rightarrow P \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_{i+1} \quad 0 \leq i < k \]
Example: Passing Property

What happens if we generalize \( \neg s = \neg x_1 \lor \neg x_2 \) at level 0 in the other way (\( \neg x_1 \))?

\[
\begin{align*}
F_0 &= I = \neg x_1 \land \neg x_2 \\
F_1 &= \neg x_1 \lor x_2
\end{align*}
\]

\[
\begin{align*}
l &\Rightarrow F_0 \\
F_i &\Rightarrow F_{i+1} & 0 \leq i < k \\
F_i &\Rightarrow P & 0 \leq i \leq k \\
F_i \land T &\Rightarrow F_{i+1}' & 0 \leq i < k
\end{align*}
\]
Example: Passing Property

Update $F_1$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \wedge T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = (\neg x_1 \lor x_2) \land \neg x_1$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Passing Property

No more CTIs in $F_1$. No counterexamples of length 2. Instantiate $F_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = (\neg x_1 \lor x_2) \land \neg x_1$

$F_2 = P = \neg x_1 \lor x_2$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Passing Property

No clauses propagate from $F_1$ to $F_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = (\neg x_1 \lor x_2) \land \neg x_1$

$F_2 = P = \neg x_1 \lor x_2$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Passing Property

Remove subsumed clauses

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = \neg x_1 \]
\[ F_2 = P = \neg x_1 \lor x_2 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \]
\[ F_i \Rightarrow P \]
\[ F_i \land T \Rightarrow F'_{i+1} \]

\[ 0 \leq i < k \]
\[ 0 \leq i \leq k \]
\[ 0 \leq i < k \]
Example: Passing Property

Does $F_2 \land T \Rightarrow P'$?

\[
\begin{align*}
F_0 &= I = \neg x_1 \land \neg x_2 \\
F_1 &= \neg x_1 \\
F_2 &= P = \neg x_1 \lor x_2
\end{align*}
\]

\[
\begin{align*}
I &\Rightarrow F_0 \\
F_i &\Rightarrow F_{i+1} \\
F_i &\Rightarrow P \\
F_i \land T &\Rightarrow F'_{i+1}
\end{align*}
\]

$0 \leq i < k$
Example: Passing Property

Found CTI $s = x_1 \land x_2$ (same as before)

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = \neg x_1 \]
\[ F_2 = P = \neg x_1 \lor x_2 \]

\[
\begin{align*}
I & \Rightarrow F_0 \\
F_i & \Rightarrow F_{i+1} \\
F_i & \Rightarrow P \\
F_i \land T & \Rightarrow F_{i+1}' \\
0 \leq i < k & \\
0 \leq i \leq k & \\
0 \leq i < k &
\end{align*}
\]
Example: Passing Property

Is $\neg s = \neg x_1 \lor \neg x_2$ inductive relative to $F_1$?

$$ F_0 = I = \neg x_1 \land \neg x_2 $$
$$ F_1 = \neg x_1 $$
$$ F_2 = P = \neg x_1 \lor x_2 $$

$$ I \Rightarrow F_0 $$
$$ F_i \Rightarrow F_{i+1} $$
$$ F_i \Rightarrow P $$
$$ F_i \land T \Rightarrow F_{i+1}' $$

$$ 0 \leq i < k $$
$$ 0 \leq i \leq k $$
$$ 0 \leq i < k $$
Example: Passing Property

No. We know it is inductive at level 0.

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = \neg x_1 \]
\[ F_2 = P = \neg x_1 \lor x_2 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \]
\[ F_i \Rightarrow P \]
\[ F_i \land T \Rightarrow F_{i+1}' \]

\[ 0 \leq i < k \]
\[ 0 \leq i \leq k \]
\[ 0 \leq i < k \]
Example: Passing Property

If generalization produces $\neg x_1$ again, the CTI is not eliminated.

\[ F_0 = I = \neg x_1 \land \neg x_2 \]
\[ F_1 = \neg x_1 \]
\[ F_2 = P = \neg x_1 \lor x_2 \]

\[
\begin{align*}
I & \Rightarrow F_0 \\
F_i & \Rightarrow F_{i+1} \\
F_i & \Rightarrow P \\
F_i \land T & \Rightarrow F'_{i+1}
\end{align*}
\]

\[
\begin{align*}
0 \leq i < k \\
0 \leq i \leq k \\
0 \leq i < k
\end{align*}
\]
Example: Passing Property

Find predecessor \( t \) of CTI \( x_1 \land x_2 \) in \( F_1 \setminus F_0 \)

\[
F_0 = I = \neg x_1 \land \neg x_2 \\
F_1 = \neg x_1 \\
F_2 = P = \neg x_1 \lor x_2
\]

\[
I \Rightarrow F_0 \\
F_i \Rightarrow F_{i+1} \\
F_i \Rightarrow P \\
F_i \land T \Rightarrow F'_{i+1}
\]

\[
0 \leq i < k \\
0 \leq i \leq k \\
0 \leq i < k
\]
Example: Passing Property

Found $t = \neg x_1 \land x_2$

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = \neg x_1$

$F_2 = P = \neg x_1 \lor x_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Passing Property

The clause $\neg t = x_1 \lor \neg x_2$ is inductive at all levels

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = \neg x_1$

$F_2 = P = \neg x_1 \lor x_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_i$

$0 \leq i < k$
Example: Passing Property

Generalization of \( \neg t = x_1 \lor \neg x_2 \) produces \( \neg x_2 \)

\[
egin{align*}
F_0 &= l = \neg x_1 \land \neg x_2 \\
F_1 &= \neg x_1 \\
F_2 &= p = \neg x_1 \lor x_2
\end{align*}
\]

\[
egin{align*}
l &\Rightarrow F_0 \\
F_i &\Rightarrow F_{i+1} \\
F_i &\Rightarrow P \\
F_i \land T &\Rightarrow F'_{i+1}
\end{align*}
\]

\[
0 \leq i < k
\]

\[
0 \leq i \leq k
\]

\[
0 \leq i < k
\]
Example: Passing Property

Update $F_1$ and $F_2$

$I \Rightarrow F_0$
$F_i \Rightarrow F_{i+1}$
$F_i \Rightarrow P$
$F_i \land T \Rightarrow F'_{i+1}$

$F_0 = I = \neg x_1 \land \neg x_2$
$F_1 = \neg x_1 \land \neg x_2$
$F_2 = (\neg x_1 \lor x_2) \land \neg x_2$

$0 \leq i < k$
$0 \leq i \leq k$
$0 \leq i < k$
Example: Passing Property

$F_1$ and $F_2$ are equivalent. Property (almost) proved

$F_0 = I = \neg x_1 \land \neg x_2$

$F_1 = \neg x_1 \land \neg x_2$

$F_2 = (\neg x_1 \lor x_2) \land \neg x_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_{i+1}$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Failing Property

No counterexamples of length 0 or 1

\[ I = \neg x_1 \land \neg x_3 \land \neg x_3 \]
\[ P = \neg x_1 \lor \neg x_2 \lor \neg x_3 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1}, \quad 0 \leq i < k \]
\[ F_i \Rightarrow P, \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_{i+1}, \quad 0 \leq i < k \]
Example: Failing Property

Does $F_1 \land T \Rightarrow P'$?

$F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3$

$F_1 = P = \neg x_1 \lor \neg x_2 \lor \neg x_3$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$, $0 \leq i < k$

$F_i \Rightarrow P$, $0 \leq i \leq k$

$F_i \land T \Rightarrow F'_{i+1}$, $0 \leq i < k$
Example: Failing Property

Found CTI $s = \neg x_1 \land x_2 \land x_3$

$F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3$

$F_1 = P = \neg x_1 \lor \neg x_2 \lor \neg x_3$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$, $0 \leq i < k$

$F_i \Rightarrow P$, $0 \leq i \leq k$

$F_i \land T \Rightarrow F_i'$, $0 \leq i < k$
Example: Failing Property

The clause $\neg s = x_1 \lor \neg x_2 \lor \neg x_3$ generalizes to $\neg x_2$ at level 0

$F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3$

$F_1 = (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land \neg x_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$ \quad $0 \leq i < k$

$F_i \Rightarrow P$ \quad $0 \leq i \leq k$

$F_i \land T \Rightarrow F_i'$ \quad $0 \leq i < k$
Example: Failing Property

No CTI left: no counterexample of length 2. $F_2$ instantiated, but no clause propagated

\[ F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3 \]
\[ F_1 = \neg x_2 \]
\[ F_2 = P = \neg x_1 \lor \neg x_2 \lor \neg x_3 \]

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \]
\[ F_i \Rightarrow P \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_{i+1} \quad 0 \leq i < k \]
Example: Failing Property

The clause \( \neg s = x_1 \lor \neg x_2 \lor \neg x_3 \) generalizes again to \( \neg x_2 \) at level 0

\[
F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3 \\
F_1 = \neg x_2 \\
F_2 = P = \neg x_1 \lor \neg x_2 \lor \neg x_3
\]

\[
l \Rightarrow F_0 \\
F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \\
F_i \Rightarrow P \quad 0 \leq i \leq k \\
F_i \land T \Rightarrow F_{i+1}' \quad 0 \leq i < k
\]
Example: Failing Property

Suppose IC3 recurs on \( t = \neg x_1 \land \neg x_2 \land x_3 \) in \( F_1 \setminus F_0 \)

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1}  \quad 0 \leq i < k \]
\[ F_i \Rightarrow P  \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_{i+1}  \quad 0 \leq i < k \]
Example: Failing Property

Clause $\neg t = x_1 \lor x_2 \lor \neg x_3$ is not inductive at level 0: the property fails.

$F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3$

$F_1 = \neg x_2$

$F_2 = P = \neg x_1 \lor \neg x_2 \lor \neg x_3$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$0 \leq i < k$

$F_i \Rightarrow P$

$0 \leq i \leq k$

$F_i \land T \Rightarrow F_i'$

$0 \leq i < k$
Example: Failing Property

Suppose now IC3 recurs on \( t = x_1 \land \neg x_2 \land x_3 \) in \( F_1 \setminus F_0 \)

\[
\begin{align*}
F_0 &= I = \neg x_1 \land \neg x_3 \land \neg x_3 \\
F_1 &= \neg x_2 \\
F_2 &= P = \neg x_1 \lor \neg x_2 \lor \neg x_3 \\
I &\Rightarrow F_0 \\
F_i &\Rightarrow F_{i+1} \quad 0 \leq i < k \\
F_i &\Rightarrow P \quad 0 \leq i \leq k \\
F_i \land T &\Rightarrow F'_i \quad 0 \leq i < k
\end{align*}
\]
Example: Failing Property

Clause $\neg t = \neg x_1 \lor x_2 \lor \neg x_3$ is inductive at level 1

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$  $0 \leq i < k$

$F_i \Rightarrow P$  $0 \leq i \leq k$

$F_i \land T \Rightarrow F'_i$  $0 \leq i < k$
Example: Failing Property

Generalization of $\neg t$ adds $\neg x_1$ to $F_1$ and $F_2$

$I \Rightarrow F_0$

$F_i \Rightarrow F_{i+1}$

$F_i \Rightarrow P$

$F_i \land T \Rightarrow F'_i$

$F_0 = I = \neg x_1 \land \neg x_3 \land \neg x_3$

$F_1 = \neg x_2 \land \neg x_1$

$F_2 = (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land \neg x_1$

$0 \leq i < k$

$0 \leq i \leq k$

$0 \leq i < k$
Example: Failing Property

Only \( t = \neg x_1 \land \neg x_2 \land x_3 \) remains in \( F_1 \setminus F_0 \)

\[
I \Rightarrow F_0 \\
F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \\
F_i \Rightarrow P \quad 0 \leq i \leq k \\
F_i \land T \Rightarrow F_{i+1}' \quad 0 \leq i < k
\]
Example: Failing Property

The same counterexample as before is found

\[ I \Rightarrow F_0 \]
\[ F_i \Rightarrow F_{i+1} \quad 0 \leq i < k \]
\[ F_i \Rightarrow P \quad 0 \leq i \leq k \]
\[ F_i \land T \Rightarrow F'_{i+1} \quad 0 \leq i < k \]
Clause Generalization

- A CTI is a cube (conjunction of literals)
  - e.g., $s = x_1 \land \neg x_2 \land x_3$

- The negation of a CTI is a clause
  - e.g., $\neg s = \neg x_1 \lor x_2 \lor \neg x_3$

- Conjoining $\neg s$ to a reachability assumption $F_i$ excludes the CTI from it

- Generalization extracts a subclause from $\neg s$ that excludes more states that are “like the CTI”
  - e.g., $\neg x_3$ may be a subclause of $\neg s$ that excludes states that, like the CTI, are not reachable in $i$ steps
  - Every literal dropped doubles the number of states excluded by a clause
  - Generalization is time-consuming, but critical to performance
Generalization

- Crucial for efficiency
- Generalization in IC3 produces a minimal inductive clause (MIC)
- The MIC algorithm is based on DOWN and UP.
- DOWN extracts the (unique) maximal subclause
- UP finds a small, but not necessarily minimal subclause
- MIC recurs on subclauses of the result of UP
Minimal Inductive Clause
Minimal Inductive Clause
Minimal Inductive Clause
Minimal Inductive Clause
Minimal Inductive Clause
Maximal Inductive Subclause (DOWN)

\[ \neg x_1 \lor x_2 \lor \neg x_3 \]
Maximal Inductive Subclause (DOWN)

\[ \neg x_1 \lor x_2 \lor \neg x_3 \]
Maximal Inductive Subclause (DOWN)

\[ x_2 \lor \neg x_3 \]
Maximal Inductive Subclause (DOWN)

\( x_2 \lor \neg x_3 \)
Maximal Inductive Subclause (DOWN)
Use of UNSAT Cores

- $\neg s \land F_i \land T \Rightarrow \neg s'$ if and only if $\neg s \land F_i \land T \land s'$ is unsatisfiable

- The literals of $s'$ are (unit) clauses in the SAT query

- If the implication holds, the SAT solver returns an unsatisfiable core

- Any literal of $s'$ not in the core can be removed from $s'$ because it does not contribute to the implication . . .

- and from $\neg s$ because strengthening the antecedent preserves the implication
Use of UNSAT Core Example

- $\neg s \land F_0 \land T \Rightarrow \neg s'$ with
  - $\neg s = \neg x_1 \lor \neg x_2$
  - $F_0 = \neg x_1 \land \neg x_2$
  - $T = (\neg x_1 \land \neg x_2 \land \neg x'_1 \land \neg x'_2) \lor \cdots$
  
- The SAT query, after some simplification, is
  - $\neg x_1 \land \neg x_2 \land \neg x'_1 \land \neg x'_2 \land x'_1 \land x'_2$

- Two UNSAT cores are
  - $\neg x'_1 \land x'_1$
  - $\neg x'_2 \land x'_2$

  from which the two generalizations we saw before follow
Clause Clean-Up

- As IC3 proceeds, clauses may be added to some $F_i$ that subsume other clauses.
- The weaker, subsumed clauses no longer contribute to the definition of $F_i$.
- However, a weaker clause may propagate to $F_{i+1}$ when the stronger clause does not.
- Weak clauses are eliminated by subsumption only between major iterations and after propagation.
More Efficiency-Related Issues

- State encoding determines what clauses are derived
- Incremental vs. monolithic
  - Reachability assumptions carry global information
  - ... but are built incrementally
- Semantic vs. syntactic approach
  - Generalization “jumps over large distances”
- Long counterexamples at low $k$
  - Typically more efficient than increasing $k$
- Consequences of no unrolling
  - Many cheap (incremental) SAT calls
- Ability to parallelize
  - Clauses are easy to exchange
Outline

1. A Short Intro to Model Checking
   - Structures
   - Properties

2. SAT Solver Interface
   - To The Solver
   - From The Solver

3. Checking Invariants
   - Bounded Model Checking
   - Interpolation
   - Proving Invariants by Induction
   - IC3: Incremental Inductive Verification

4. Progress Properties and Branching Time
   - Bounded Model Checking
   - Incremental Inductive Verification (FAIR and $k$-Liveness)
   - Model Checking CTL
BMC: Translation from LTL

- Various techniques have been devised to translate an LTL formula $\varphi$ into a propositional formula that expresses the constraints on a path that is a model of $\neg \varphi$. For instance:

$$\left[ \neg \mathsf{F} \mathsf{G} \neg p \right] = \bigvee_{0 \leq l \leq k} (T(\bar{x}_k, \bar{x}_l) \land \bigvee_{l \leq i \leq k} p(\bar{x}_i))$$

- $k$-induction can be extended to provide a termination criterion.
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BMC: Liveness to Safety

- Checking progress properties requires cycle detection
- Augment model with shadow register
- The augmented model can nondeterministically save a snapshot of the current state in the shadow register
- If a state is subsequently reached that is identical to the one saved, a cycle has been detected
- Constraints can be added to make sure the cycle is an accepting one
- With this transformation an invariant checker suffices for all LTL properties
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FAIR: Finding Reachable Fair Cycles

- Check language nonemptiness of the composition of structure $S$ and \textit{generalized} Büchi automaton for $\neg \varphi$
- Generalized means that multiple acceptance conditions (aka fairness constraints may be given: each must be satisfied
- FAIR (Bradley et al. [2011]) looks for a reachable fair cycle
- The search for a cycle is decomposed into several reachability queries
  - Each reachability query is a call to IC3
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A counterexample to a progress property is a lasso-shaped path that satisfies fairness constraints.

A lasso’s cycle is contained in a strongly connected component (SCC) of the state graph.

A nonempty set of states is SCC-closed if every SCC is either contained in it or disjoint from it.

A partition of the states into SCC-closed sets is a coarser partition than the SCC partition; hence, …

Every cycle of a graph is contained in some SCC-closed set.

Maintain a partition of the states into SCC-closed set.

Refine it until a reachable fair cycle is found or none is proved to exist.
Strongly Connected Components

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FAIR: Finding Reachable Fair Cycles

Reduce search for reachable fair cycle to a set of safety problems:

- **Skeleton:**

  States of skeleton together satisfy all fairness constraints.

- **Task:** Connect states to form lasso.
Reach Queries

Each connection task is a reach query.

- **Stem query**: Connect initial condition to a state:

- **Cycle query**: Connect one state to another:

(To itself if skeleton has only one state.)
Witness to Nonemptiness

If all queries are answered positively:

Witness to nonemptiness of $C$. 
Global Reachability

If a stem query is answered negatively: new inductive global reachability information.

- Constrains subsequent selection of skeletons.
- Constrains subsequent reach (stem and cycle) queries.
- Improve proof by strengthening (using ideas from IC3).
Barriers: Discovering SCC-Closed Sets

If a cycle query is answered negatively: new information about SCC structure of state graph.

- **Inductive** proof: “one-way barrier”
- Each “side” of the proof is SCC-closed.
- Constrains subsequent selections of skeletons: all states on one side.
Example: Empty Language
Example: Empty Language
Example: Empty Language

stem query produces $x_1 \lor \neg x_2$
Example: Empty Language
Example: Empty Language

states satisfy $x_1 \lor \neg x_2$
Example: Empty Language

states satisfy $x_1 \lor \neg x_2$

stem query passes
Example: Empty Language

$sk2 \quad 101 \quad 110$

states satisfy

$x_1 \lor \neg x_2$

reach($S$, $(x_1 \lor \neg x_2)$, $s_0, s_1$) passes
Example: Empty Language

states satisfy \( x_1 \lor \neg x_2 \)

reach\((S, (x_1 \lor \neg x_2), s_1, s_0)\) produces \( x_2 \)
Example: Empty Language

states satisfy
\[ x_1 \lor \neg x_2 \]

because \[ x_1 \land x_2 \land \neg x_3 \Rightarrow x_2 \ldots \]
Example: Empty Language

states satisfy
\[ x_1 \lor \neg x_2 \]

and \( x_2 \land (x_1 \lor \neg x_2) \land T \Rightarrow x'_2 \)
Example: Empty Language
Example: Empty Language

states satisfy $(x_1 \lor \neg x_2) \land \neg x_2$
Example: Empty Language

states satisfy
\((x_1 \lor \neg x_2) \land \neg x_2\)

stem query passes
Example: Empty Language

\[ s_0 \quad s_1 \]
\[ sk3 \quad 101 \quad 100 \]

states satisfy
\[ (x_1 \lor \neg x_2) \land \neg x_2 \]

reach\((S, (x_1 \lor \neg x_2) \land \neg x_2, s_0, s_1)\) produces \(x_2 \lor x_3\)
Example: Empty Language

no skeletons left
Example: Single-State Skeleton
Example: Single-State Skeleton

$sk_1 \quad s_0 \quad s_1$

$000 \quad 001 \quad 010 \quad 011$

$100 \quad 101 \quad 110 \quad 111$

$s_0 = s_1$
Example: Single-State Skeleton

stem query passes
Example: Single-State Skeleton

$sk1 \quad s_0 \quad s_1$

$000 \quad 001 \quad 010 \quad 011$

$100 \quad 101 \quad 110 \quad 111$

reach($S, \top, \text{post}(S, s_0), s_0$) produces $x_1 \land x_2$

and $(\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3)$
Example: Single-State Skeleton
Example: Single-State Skeleton

$$s_0$$ 001 100

$$s_1$$ 001 100

states satisfy

$$\neg x_1 \lor \neg x_2 \land \neg x_1 \lor \neg x_3$$
Example: Single-State Skeleton

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>$s_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>001</td>
<td>100</td>
</tr>
</tbody>
</table>

States satisfy:

$$(\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3)$$

Stem query passes
Example: Single-State Skeleton

states satisfy

\((\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3)\)

reach\((S, (\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3), s_0, s_1)\) produces \(x_2 \lor x_3\)
Example: Single-State Skeleton
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sk3 \quad 001 \quad 011

states satisfy

\((\neg x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_3) \land (x_2 \lor x_3)\)
Example: Single-State Skeleton

states satisfy
\((\neg x_1 \lor \neg x_2) \land (
\neg x_1 \lor \neg x_3) \land (x_2 \lor x_3)\)

stem query passes
Example: Single-State Skeleton

\[ s_0 \quad s_1 \]

\[ \text{sk3} \quad 001 \quad 011 \]

states satisfy

\[ \neg x_1 \lor \neg x_2 \land \neg x_1 \lor \neg x_3 \land x_2 \lor x_3 \]

\[ \text{reach}(S, \neg x_1 \lor \neg x_2) \land \neg x_1 \lor \neg x_3) \land (x_2 \lor x_3, s_0, s_1) \text{ produces } \neg x_2 \]
Example: Single-State Skeleton
Example: Single-State Skeleton

\[ s_0 \quad s_1 \]

\[ \text{sk4} \quad 010 \quad 011 \]

states satisfy
\[
(\neg x_1 \lor \neg x_2) \land \\
(\neg x_1 \lor \neg x_3) \land \\
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Example: Single-State Skeleton

states satisfy

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(\neg x_1 \lor \neg x_3) \land 
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stem query produces $x_1 \lor \neg x_2$
Example: Single-State Skeleton

no skeletons left

Diagram showing transitions between states: 000 -> 001, 001 -> 010, 010 -> 011, 100 -> 101, 101 -> 110, 110 -> 111.
Persistent Signals

- Signal $p$ is **persistent** in structure $S$ if

  $$S \models G(p \to Xp)$$

  or

  $$S \models G(\neg p \to X\neg p)$$

- Checking for persistence by a SAT check:

  $$p \land T \Rightarrow p'$$

  $$\neg p \land T \Rightarrow \neg p'$$
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Barriers from Persistent Signals

- Signals may be persistent under assumptions
  - Another signal is persistent
  - Another signal has a given value

- A persistent signal defines a barrier
- One side of the barrier may have no skeletons
- Then the persistent signal may be assumed to have a fixed value
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Slice’n’Dice

Property holds if $F G \neg f$. 

\[
x_1 x_0, f
\]

\[
\begin{align*}
&\quad 00, 1 \\
&\rightarrow 01, 1 \\
&\rightarrow 10, 1 \\
&\rightarrow 11, 1 \\
&\quad 11, 0 \\
&\quad 10, 0 \\
&\quad 01, 0 \\
&\quad 00, 0
\end{align*}
\]
Slice’n’Dice

Property holds if $F \mathcal{G} \neg f$.

\[ x_1 x_0, f \]

\[
\begin{array}{cccc}
00, 1 & \rightarrow & 01, 1 & \rightarrow & 10, 1 & \rightarrow & 11, 1 \\
11, 0 & \leftarrow & 10, 0 & \leftarrow & 01, 0 & \leftarrow & 00, 0
\end{array}
\]
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Property holds if $F \mathcal{G} \neg f$. 

\[ x_1 x_0, f \]

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10, 1 & \rightarrow 11, 1
\end{align*} \]
Slice’n’Dice

Property holds if $F G \neg f$. 

```
x_1 x_0, f
```

```
00, 1 --01, 1 --10, 1 --11, 1
```

```
11, 0 --10, 0 --01, 0 --00, 0
```

Key Insights

- Inductive assertions describe SCC-closed sets.
- Arena: Set of states all on the same side of each barrier.
- Unlike previous symbolic methods:
  
  *Barrier constraints on the transition relation combined with the over-approximating nature of IC3 enable the simultaneous (symbolic) consideration of all arenas.*

- A proof can provide information about many arenas even though the motivating skeleton comes from one arena.
Methodological Parallels with IC3

<table>
<thead>
<tr>
<th>IC3</th>
<th>FAIR</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Seed:</strong></td>
<td>CTI</td>
</tr>
<tr>
<td><strong>Lemma:</strong></td>
<td>Inductive clause</td>
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<tr>
<td></td>
<td>Global reachability proof</td>
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<tr>
<td></td>
<td>One-way barrier</td>
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<tr>
<td></td>
<td><em>Relative to previously discovered lemmas.</em></td>
</tr>
<tr>
<td><strong>CEX:</strong></td>
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</tr>
<tr>
<td></td>
<td>Connected skeleton</td>
</tr>
<tr>
<td></td>
<td><em>Discovery guided by lemmas. Not minimal.</em></td>
</tr>
<tr>
<td><strong>Proof:</strong></td>
<td>Inductive strengthening</td>
</tr>
<tr>
<td></td>
<td>All arenas skeleton-free</td>
</tr>
<tr>
<td></td>
<td><em>Sufficient set of lemmas.</em></td>
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</table>
$k$-Liveness (Claessen and Sörensson [2012])

- If property holds in $S$, then $S \models F G \neg p$
  - $p$ holds finitely many times
- Approximate with sequence of safety properties
  - $p$ is never true
  - $p$ holds at most once
  - $p$ holds at most twice \ldots
  - $p$ holds at most $k$ times \ldots
- If any property in the sequence holds, so does $F G \neg p$
- If $S$ is finite-state, then $S \models F G \neg p$ holds only if there is $k$ such that $p$ holds at most $k$ times
- $k$-liveness is in practice a semi-decision procedure
  - Interleave BMC calls to check whether property fails
- For each value of $k$, IC3 decides whether safety property holds
  - In principle, any safety model checker would do
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- Each subcircuit *absorbs* one occurrence of $p$
- Increasing $k$ means adding another instance of the subcircuit
- This solution works well with an incremental safety solver
- General approach relies on representing property as a universal co-Büchi word automaton ([Filiot et al. 2009], bounded synthesis)
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  - They are used to constrain the transition relation
  - More likely to prove the property at a lower $k$

Without constraints:
- $FG \neg f$ proved at $k = 5$

With constraints $f \land (x_1 \leftrightarrow x'_1) \land (x_0 \leftrightarrow x'_0)$:
- No infinite paths: $FG \neg f$ proved at $k = 0$
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- No infinite paths: $FG\neg f$ proved at $k = 0$
IICTL: Incremental Inductive CTL Model Checking

- Task-directed strategy
- Maintains upper and lower bounds on states satisfying each subformula
- States in between the bounds are undecided
- Typically don’t need to decide all states to decide the property (Traditional symbolic CTL algorithms do)
- Decide states by executing appropriate query:
  - EX: SAT query
  - EU: Safety model checker (e.g., IC3)
  - EG: Fair cycle finder (e.g., FAIR)
- Generalizing decisions (proofs or counterexamples) to other states and refining the bounds
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$
Property: $\text{AG EF } p = \neg EF \neg EFp$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

\[
\begin{array}{c}
\neg \\
\text{EF} \\
\neg \\
\text{EF} \\
p
\end{array}
\]
Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

\[
\begin{align*}
\neg & \quad [\text{EF } \neg \text{EF } p] \supseteq \text{initial states?} \\
\text{EF} & \quad [\text{EF } \neg \text{EF } p] \\
\neg & \quad [\text{EF } p] \\
\text{EF} & \quad [\text{EF } p] \\
\quad p & \quad [p]
\end{align*}
\]
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

Diagram:

- $\neg \text{EF} \neg \text{EF} p$ initial states?
  - Yes: Property holds
  - No: Property fails
- $\text{EF} \neg \text{EF} p$
- $\neg \text{EF} p$
- $\text{EF} p$
- $p$
IICTL Example

Property: AG EF p = ¬EF¬EFp
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: AG EF p = ¬EF ¬EF p

![Diagram of IICTL Example]
IICTL Example

Property: $\mathrm{AG\ EF\ } p = \neg\mathrm{EF\ \neg\mathrm{EF\ }p}$
IICTL Example

Property: AG EF $p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EFp$
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

Diagram:

- $0 \vdash \psi_0$
- $1 \vdash \psi_1$
- $2 \neg \psi_2 \neg p$
- $3 \vdash \psi_3 \text{EF } p$
- $4 \text{EF } p$
- $p \vdash \psi_4$

$I \land \neg \text{U}_0$?
IITL Example

Property: $\text{AG EF } p = \neg EF \neg EFp$

$I \land \neg U_0$? No: Property fails
IICTL Example

Property: $AG \ EF \ p = \neg EF \neg EFp$

$I \land \neg U_0? $ Yes

Diagram:

0 $\top$
1 $\top$
2 $\neg p$
3 $\top$
4 $p$
5 $p$
6 $p$
7 $\psi_0$
8 $\psi_1$
9 $\psi_2$
10 $\psi_3$
11 $\psi_4$

$\top, \bot$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

$I \land \neg U_0$? Yes

$I \land \neg L_0$?
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

$I \land \neg U_0$? Yes

$I \land \neg L_0$? Yes: Property holds
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

$I \land \neg U_0$? Yes
$I \land \neg L_0$? No
IICTL Example

Property: $AG\ EF\ p = \neg EF\ \neg EF\ p$

$I \land \neg U_0$? Yes

$I \land \neg L_0$? No $s \not\models L_0$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

\[
\begin{align*}
\psi_0 & \quad I \land \neg U_0? \quad \text{Yes} \quad s \models U_0 \\
\psi_1 & \quad I \land \neg L_0? \quad \text{No} \quad s \not\models L_0
\end{align*}
\]
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EF p$

$s$ is undecided for node 0

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$
IICTL Example

Property: $\text{AG EF } p = \neg\text{EF} \neg\text{EF} p$

$I \land \neg U_0$? Yes $s \models U_0$
$I \land \neg L_0$? No $s \not\models L_0$

$s$ is undecided for node 0
Property: $\text{AGEF } p = \neg\text{EF} \neg p$

$s\models U_0$? Yes
$s\models L_0$? No

$s$ is undecided for node 0

$\psi_0 \psi_1 \psi_2 \psi_3 \psi_4$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

1. $I \land \neg U_0$? Yes $s \models U_0$
2. $s$ is undecided for node 0
3. $I \land \neg L_0$? No $s \not\models L_0$
4. $s \models \psi_1$?
ICTL Example

Property: \( \text{AG EF } p = \neg \text{EF } \neg \text{EF } p \)

\[ \begin{align*}
\text{I } \land \neg U_0 & \text{? Yes } s \models U_0 \\
\text{I } \land \neg L_0 & \text{? No } s \not\models L_0 \\
\end{align*} \]

\( s \models \psi_1 \iff s \models \text{EF } \psi_2 ? \)
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EFp}$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

$s \models \psi_1 \iff s \models \text{EF } \psi_2$? $\iff$ can $s$ reach $\psi_2$?
IICTL Example

Property: $\text{AG } EF \ p = \neg EF \neg EF \ p$

$s \models \psi_0 \iff s \models \psi_2 \iff \text{can s reach } \psi_2$?

$s \models \psi_1 \iff s \models EF \psi_2 \iff \text{can s reach } \psi_2$?

$s \models \psi_1$? $\iff s \models EF \psi_2$? $\iff$ can s reach $\psi_2$?

$s$ is undecided for node 0

$s \models U_0$?

$s \models L_0$?
IICTL Example

Property: $\forall G \exists F p = \neg EF \neg EF p$

$I \land \neg U_0$? Yes $s \models U_0$
$I \land \neg L_0$? No $s \not\models L_0$

$s \models \psi_1$? $\iff$ $s \models EF \psi_2$? $\iff$ can $s$ reach $\psi_2$?

Can $s$ reach $L_2$? Yes: $s$ can also reach $\psi_2$
Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

$l \land \neg U_0$? Yes $s \models U_0$

$l \land \neg L_0$? No $s \not\models L_0$

$s \models \psi_1 \iff s \models \text{EF } \psi_2\iff$ can $s$ reach $\psi_2$?

can $s$ reach $L_2$? No
**IICTL Example**

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

$I \land \neg U_0$? Yes  $s \models U_0$

$I \land \neg L_0$? No  $s \not\models L_0$

Can $s$ reach $U_2$?

$s \models \psi_1 \iff s \models \text{EF } \psi_2$?  $\iff$ can $s$ reach $\psi_2$?

Can $s$ reach $L_2$? No

$s$ is undecided for node 0
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF}p$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

$s$ is undecided for node 0

Can $s$ reach $U_2$? No: $s$ cannot reach $\psi_2$

$s \models \psi_1 \iff s \models \text{EF } \psi_2 \iff$ can $s$ reach $\psi_2$?

Can $s$ reach $L_2$? No
**IICTL Example**

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

- $I \land \neg U_0$? Yes $s \models U_0$
- $I \land \neg L_0$? No $s \not\models L_0$
- Can $s$ reach $U_2$? Yes $s \models \psi_1 \iff s \models \text{EF } \psi_2$?
- Can $s$ reach $L_2$? No

$s$ is *undecided* for node 0
IICTL Example

Property: AG EF p = ¬EF ¬EFp

\[ I \land \neg U_0? \quad \text{Yes} \quad s \models U_0 \]
\[ I \land \neg L_0? \quad \text{No} \quad s \not\models L_0 \]

\[ \text{can } s \text{ reach } U_2? \quad \text{Yes} \quad t \models U_2 \]
\[ s \models \psi_1? \iff s \models EF \psi_2? \iff \text{can } s \text{ reach } \psi_2? \]
\[ \text{can } s \text{ reach } L_2? \quad \text{No} \]
**IICTL Example**

Property: $\text{AG } EF \, p = \neg EF \neg EF \, p$

- Initial state $s$.
- $I \land \neg U_0$? Yes $s \models U_0$.
- $s$ is undecided for node 0.
- $I \land \neg L_0$? No $s \not\models L_0$.
- Can $s$ reach $U_2$? Yes $t \models U_2$.
- $s \models \psi_1 \iff s \models EF \psi_2$? $\iff$ can $s$ reach $\psi_2$?
- Can $s$ reach $L_2$? No $t \not\models L_2$. 

Diagram:

- Node 0: $s_0 \models \psi_0$.
- Node 1: $s_1 \models \psi_1$.
- Node 2: $s_2 \models \neg p \models \psi_2$.
- Node 3: $s_3 \models \psi_3$.
- Node 4: $s_4 \models \psi_4$.

Nodes 1 and 3 are labeled with EF.
IICTL Example

Property: $\text{AG} \ EF \ p = \neg EF \neg EFp$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

can $s$ reach $U_2$? Yes $t \models U_2$

t is undecided for node 2

can $s$ reach $L_2$? No $t \not\models L_2$

$s$ is undecided for node 0
IICTL Example

Property: $\text{AG EF } p = \neg EF \neg EFp$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

can $s$ reach $U_2$? Yes $t \models U_2$

can $s$ reach $L_2$? No $t \not\models L_2$
IICTL Example

Property: $\text{AG EF } p = \neg \text{EF } \neg \text{EF } p$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

Can $s$ reach $U_2$? Yes $t \models U_2$

Can $s$ reach $L_2$? No $t \not\models L_2$

$s$ is undecided for node 0

$t$ is undecided for node 2
IICTL Example

Property: $\text{AG} \, EF \, p = \neg EF \, \neg EFp$

- $I \land \neg U_0$? Yes $s \models U_0$
  - $s$ is undecided for node 0
- $I \land \neg L_0$? No $s \not\models L_0$
  - can $s$ reach $U_2$? Yes $t \models U_2$
    - $t$ is undecided for node 2
  - can $s$ reach $L_2$? No $t \not\models L_2$

- can $t$ reach $L_4$ (or $U_4$)?
**IICTL Example**

Property: $\text{AG EF } p = \neg \text{EF} \neg \text{EF} p$

- $s$ is undecided for node 0
- $s \models U_0$? Yes
- $s \not\models L_0$? No
- $s$ is undecided for node 0

- $t$ is undecided for node 2
- can $s$ reach $U_2$? Yes
- $t \models U_2$? Yes
- $t$ is undecided for node 2
- can $s$ reach $L_2$? No
- $t \not\models L_2$? Yes

- $I \land \neg U_0$? Yes
- $I \land \neg L_0$? No

- can $t$ reach $L_4$ (or $U_4$)?
**IICTL Example**

Property: $\text{AG } EF\ p = \neg EF\neg EF\ p$

- $s$ is undecided for node 0
- $l \wedge \neg U_0$? Yes $s \models U_0$
- $l \wedge \neg L_0$? No $s \not\models L_0$
- Can $s$ reach $U_2$? Yes $t \models U_2$
- $t$ is undecided for node 2
- Can $s$ reach $L_2$? No $t \not\models L_2$

Can $t$ reach $L_4$ (or $U_4$)?
IICTL Example

Property: $AG\ EF\ p = \neg EF\ \neg EFp$

$I \land \neg U_0$? Yes $s \models U_0$

$I \land \neg L_0$? No $s \not\models L_0$

Can $s$ reach $U_2$? Yes $t \models U_2$

Can $s$ reach $L_2$? No $t \not\models L_2$

Can $t$ reach $L_4$ (or $U_4$)?
**IICTL Example**

Property: AG EF \( p = \neg EF \neg EF p \)

\[ \begin{align*}
0 & : I \land \neg U \text{? Yes } s \models U \\
1 & : I \land \neg L \text{? No } s \not\models L \\
2 & : \neg p \land \neg t \\
3 & : I \\
4 & : p \lor t \\
5 & : p
\end{align*} \]

- \( s \) is undecided for node 0
- \( t \) is undecided for node 2
- \( s \) can reach \( U_2 \)? Yes \( t \models U_2 \)
- \( s \) can reach \( L_2 \)? No \( t \not\models L_2 \)
- \( I \land \neg U_0 \text{? Yes } s \models U_0 \)
- \( I \land \neg L_0 \text{? No } s \not\models L_0 \)
- \( \neg p \models \)
- \( \bot \models \)
- \( \top \models \)
- \( \bot \models \)
- \( \neg \models \)

\[ \begin{align*}
\psi_0 & \}
\psi_1 & \}
\psi_2 & \}
\psi_3 & \}
\psi_4 & \}
\end{align*} \]
IICTL Algorithm

1. Construct the parse-graph of the formula
2. Initialize bounds
3. Are all initial states in lower bound of root node?
   Yes: property holds
4. Is any of the initial states not in upper bound of root?
   Yes: property fails
5. There is an *undecided* state $s$. Decide $s$ recursively and generalize.
6. Repeat step 3